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Positively Invariant Regions for Strongly Coupled Reaction–Diffusion Systems with a Balance Law

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1. INTRODUCTION

This paper will present a simple method which sometimes may enable one to find positively invariant regions for reaction–diffusion systems of the type

$$\frac{\partial u_i(x,t)}{\partial t} = \sum_{j=1}^{m} a_{ij} L u_j(x,t) + F_i(x,t,U), \qquad x \in \Omega \text{ and } t > 0, \quad (1)$$

with initial conditions

$$u_j(x,0) = u_j^0(x)$$

and boundary conditions

$$\frac{\partial u_j(x,t)}{\partial \nu} = G_i(x,t,U). \quad (2)$$

Here Ω is a bounded open set in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and

$$U := (u_1, u_2, \ldots, u_m)$$

is a function from $\Omega \times \mathbf{R}_+$ into $\mathbf{R}^m$. $A := (a_{ij})$ is an $m \times m$ constant matrix. F is a continuous function from $\overline{\Omega} \times [0,\infty) \times \mathbf{R}^m$ into $\mathbf{R}^m$, G is a continuous function from $\overline{\Omega} \times \mathbf{R}^m$ into $\mathbf{R}^m$, and $\partial/\partial\nu$ represents the outward conormal derivative at $\partial\Omega$ associated with the uniformly elliptic operator L.

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This work was motivated by recent results of Kanel and Kirane [3, 4] who have studied the strongly coupled system

$$u_t = a_{11}\Delta u + a_{12}\Delta v - uf(v) \tag{3}$$

$$v_t = a_{21}\Delta u + a_{22}\Delta v + uf(v), \tag{4}$$

where the coefficients a_{ij} are nonnegative constants. Systems such as these and their quasilinear generalizations have received much attention because they arise in several chemical and biological models. The reader is referred to the expository article by Ni [6] and to further references that may be found in [3] and [4].

2. POSITIVELY INVARIANT SETS FOR WEAKLY COUPLED SYSTEMS

We will first state a result from [5] on positively invariant regions, but since we will not need it in its full generality we will only state a special case of Theorem 16 from that paper and under much simplified hypotheses. In particular, we will avoid tangential derivatives in the boundary conditions and we will impose simple regularity requirements on the coefficients.

Let Ω be a bounded domain in $\mathbf{R}^n$ whose boundary is of class C^2. In this region we will define the formal partial differential operators

$$L_k u = -D_i a_k^{ij}(x) D_j u + b_k^i D_i u + c_k u, \tag{5}$$

where $k = 1, 2, \dots, m$. All coefficients are assumed to belong to $L^\infty(\Omega)$ and summation over i and j is implied. We also define the boundary operators

$$B_k u = \nu_i a_k^{ij} D_j u + \sigma_k u,$$

where the coefficients are assumed to belong to $L^\infty(\partial\Omega)$ and $a_k^{ij} \in C^1(\overline{\Omega})$. We assume that the operators L_k are uniformly elliptic. Mixed boundary conditions will be allowed. For each $1 \le k \le m$ we assume that $\partial\Omega = \Delta_k \cup \Gamma_k$ where $\Gamma_k := \partial\Omega \setminus \Delta_k$ and Δ_k is an open subset of $\partial\Omega$ whose boundary in $\partial\Omega$ has zero $(n-1)$-dimensional Lebesgue measure. We use the Δ and Γ to denote the cartesian products

$$\Delta := \Delta_1 \times \Delta_2 \times \cdots \times \Delta_m \qquad \text{and} \qquad \Gamma := \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_m.$$

Let $H^1(\Omega)$ be the Sobolev space of real $L^2(\Omega)$ functions with derivatives in $L^2(\Omega)$. We use the notation $\mathscr{H}^1(\Omega) := (H^1(\Omega))^m$. Similarly, we denote $C(\Omega)^m$ by $\mathscr{C}(\Omega)$, $(L^p(\Omega))^m$ by $\mathscr{L}^p(\Omega)$, and $(H^s(\Omega))^m$ by $\mathscr{H}^s(\Omega)$.

All spaces are assumed real.

Consider the elliptic system

$$L_k u_k(x) = f_k(x) \qquad \text{for } x \in \Omega, \tag{6}$$

$$B_k u_k(x) = g_k(x) \qquad \text{for } x \in \Delta_k, \tag{7}$$

$$u_k(x) = \theta_k(x) \qquad \text{for } x \in \Gamma_k. \tag{8}$$

With this problem we can associate a bilinear form on $\mathscr{H}^1(\Omega \times \mathscr{H}^1(\Omega)$ in the usual manner:

$$\begin{aligned} A(U,V) := \sum_{k=1}^{m} \Big\{ \big(a_k^{ij} (D_j u_k)(D_i v_k) \big)_\Omega + \big(b_k^i D_i u_k, v_k \big)_\Omega \\ + (c_k u_k, v_k)_\Omega + (\sigma_k u_k, v_k)_{\Delta_k} \Big\} \\ = \sum_{k=1}^{m} \big\{ (f_k, v_k) + (g_k, v_k)_{\Delta_k} \big\}, \end{aligned} \tag{9}$$

where $(\ ,\ )_\Omega$ denotes the usual inner product on $L^2(\Omega)$, $(\ ,\ )_{\Delta_k}$ denotes the $\mathscr{H}^{1/2}(\Omega)$–$\mathscr{H}^{-1/2}(\Omega)$ duality, and $(\ ,\ )$ denotes the $\mathscr{H}^1(\Omega)$–$\mathscr{H}^1(\Omega)^*$ duality.

DEFINITION. Suppose that $F \in \mathscr{H}^1(\Omega)^*$, the dual space of $\mathscr{H}^1(\Omega)$, and $G \in \mathscr{H}^{-1/2}(\Omega)$ and $\Theta \in \mathscr{H}^1(\Omega) \cap \mathscr{L}^\infty(\Omega)$. Let $\mathscr{H}_\Delta^1(\Omega)$ denote the closure in $\mathscr{H}^1(\Omega)$ of all functions in $\mathscr{H}^1(\Omega)$ whose kth component is zero on an open neighborhood of Γ_k for $k = 1, 2, \dots, m$. We say that U is a *strong solution* of the above elliptic system if $U - \Theta \in \mathscr{H}_\Delta^1(\Omega)$ and

$$A(U,V) = (F,V) + (G,V)_\Delta$$

for all $V \in \mathscr{H}_\Delta^1(\Omega)$ (or, equivalently, for all $V \in \mathscr{C}^\infty(\Omega)$ whose kth component is zero on a neighborhood of Γ_k for $k = 1, 2, \dots, m$).

In this definition we have made some obvious simplifications in the notation such as omitting the reference to the trace operator γ_0: $H^1(\Omega) \to H^{1/2}(\partial\Omega)$ that gives us the restriction of $H^1(\Omega)$ functions to the boundary.

Consider the parabolic system

$$\partial u_k / \partial t + L_k u_k(x) = f_k(x, t, U) \qquad \text{for } (x,t) \in \Omega \times (0,T) \tag{10}$$

$$B_k u_k(x) = g_k(x, U) \qquad \text{for } (x,t) \in \Delta_k \times (0,T) \tag{11}$$

$$u_k(x) = \theta_k(x) \qquad \text{for } (x,t) \in \Gamma_k \times (0,T) \tag{12}$$

$$u_k(x, 0) = u_k^0(x) \tag{13}$$

DEFINITION. We say that $U : [0, T) \to \mathscr{L}^2(\Omega)$ is a *strong solution* of the above system if:

(a) U is continuous on $[0, T)$ and $U(0) = U^0$.

(b) U is absolutely continuous on compact subsets of $(0, T)$.

(c) U is differentiable a.e. on $(0, T)$ and for each t where the derivative exists U is a strong solution of (10)–(13) regarded as an elliptic system.

DEFINITION. A subset $\mathscr{S} \subset \mathscr{L}^2(\Omega)$ is called a *positively invariant set* (or more simply an *invariant set*) if whenever U is a strong solution of the above parabolic system with $U(0) \in \mathscr{S}$ then $U(t) \in \mathscr{S}$ for all $t \in (0, T)$.

We assume the functions $f_k(x, t, U)$ and $g_k(x, U)$ are continuous. Moreover, we assume that $f_k(s, t, U)$ and $g_k(x, U)$ are Lipschitz continuous in t and U.

If there exists an invariant set $\mathscr{S}$ such that

$$|\mathscr{S}| := \sup\{\|w_k(x)\|_{L^\infty(\Omega)} : 1 \le k \le m, (w_1, w_2, \dots, w_m) \in \mathscr{S}\} < \infty,$$

then there exists a strong solution, $U(t)$, on $[0, \infty)$ whenever $U(0) \in \mathscr{S}$ (see Theorem 16 in [5]). To find invariant sets we state a simplified version of Corollary 18 of [5], but first we need to introduce some notation. If w and v are Lebesgue measurable functions on some common domain then $w \le v$ (or equivalently $v \ge w$) means that $w(x) \le v(x)$ almost everywhere. If $W := (w_1, w_2, \dots, w_m)$ and $V := (v_1, v_2, \dots, v_m)$ where all the entries are measurable functions on a common domain, then we write $W \le V$ or $V \ge W$ if these relations hold for all corresponding components. These order relations can be extended to duals of Sobolev spaces by defining $\alpha \ge \beta$ if $(\alpha, u) \ge (\beta, u)$ for all $u \ge 0$. By $[\Phi, \Psi]$ we will mean the closed order interval between Φ and Ψ.

We will need notation to denote the faces of the order interval $[\Phi, \Psi]$:

$$\Phi_k := \{U \in \mathscr{H}^1(\Omega) \cap \mathscr{L}^\infty(\Omega) \cap \mathscr{C}(\Omega) : u_k = \phi_k \text{ and } \phi_i \le u_i \le \psi_i \text{ for all } i \ne k\}$$

$$\Psi_k := \{U \in \mathscr{H}^1(\Omega) \cap \mathscr{L}^\infty(\Omega) \cap \mathscr{C}(\Omega) : u_k = \psi_k \text{ and } \phi_i \le u_i \le \psi_i \text{ for all } i \ne k\}$$

THEOREM 1. *Let* $\Phi, \Psi \in \mathscr{H}^1(\Omega) \cap \mathscr{L}^\infty(\Omega) \cap \mathscr{C}(\Omega)$, *with* $\Phi \le \Psi$. *Suppose that* $\Phi \le \Theta \le \Psi$ *and for each* k

$$\begin{gathered} L_k u_k \le f_k(x,t,U)\ \forall U \in \Phi_k \qquad \text{and} \\ L_k u_k \ge f_k(x,t,U)\ \forall U \in \Psi_k \text{ in } \Omega \times [0,T) \end{gathered} \tag{14}$$

$$\begin{gathered} B_k u_k \le g_k(x,U)\ \forall U \in \Phi_k \qquad \text{and} \\ B_k u_k \ge g_k(x,U)\ \forall U \in \Psi_k \text{ in } \partial\Omega \times [0,T), \end{gathered} \tag{15}$$

then $[\Phi, \Psi]$ *is an invariant set. For any initial condition in* $[\Phi, \Psi]$ *there exists a unique, global, strong solution.*

A special case of the above result is the so-called invariant rectangle result. In this case one takes the functions Φ and Ψ to be constant and (14)–(15) will no longer contain derivatives. Suppose that $\phi_j \equiv \alpha_j \in \mathbf{R}$ and $\psi_j \equiv \beta_j \in \mathbf{R}$, $1 \le j \le m$. We will allow ourselves to abuse the terminology somewhat and to refer to the parallelipiped

$$\{u : \alpha_j \le u_j \le \beta_j \text{ for } 1 \le j \le m\} \subset \mathbf{R}^m$$

as the invariant set.

Note. The above theorem also holds if we replace any of the functions ϕ_j by $-\infty$ or any of the functions ψ_j by $+\infty$ provided we use the convention $L_j(\pm\infty) := \pm\infty$ and $B_j(\pm\infty) := \pm\infty$.

3. INVARIANT SETS FOR STRONGLY COUPLED SYSTEMS

Suppose the matrix A in Eq. (1) has a full set of left eigenvectors $\{\Upsilon_j\}$ corresponding to the simple eigenvalues $\{\lambda_j\}$. Let Υ be the $m \times m$ matrix formed with the rows Υ_j and let $\Lambda := \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. We make the change of variables $W = \Upsilon U$. In the new coordinates the system becomes

$$\frac{\partial W}{\partial t} = \Lambda L W + \Upsilon F(\Upsilon^{-1} W, x, t)$$

with boundary conditions

$$\frac{\partial W}{\partial \nu} + \Upsilon G(\Upsilon^{-1} W, x, t). \tag{16}$$

If this system has a positively invariant set $\mathscr{S}$ then the original system has the positively invariant set $\Upsilon^{-1}(\mathscr{S})$. This idea was also used by Redlinger in [7] to extend the fundamental results of Chueh *et al.* [1] to strongly coupled systems. It is not our aim in this paper to obtain an extremely general result and so we restrict ourselves, like Kanel and Kirane [4] to two-component systems with constant principal coefficients,

$$\begin{aligned} u_t + a_{11}Lu + a_{12}Lv &= -f_1(x,t,u,v) \\ v_t + a_{21}Lu + a_{22}Lv &= f_2(x,t,u,v), \end{aligned} \tag{17}$$

and with boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial \nu} &= -f_3(x,t,u,v) \text{ on } \Delta \qquad \text{and} \qquad u = \theta_1 \text{ on } \Gamma, \\ \frac{\partial v}{\partial \nu} &= f_4(x,t,u,v) \text{ on } \Delta \qquad \text{and} \qquad v = \theta_2 \text{ on } \Gamma, \end{aligned} \tag{18}$$

where L is a uniformly elliptic operator, as in (5), $\partial/\partial\nu$ represents the outward conormal derivative and $\Delta \cup \Gamma = \partial\Omega$ is a disjoint union. We assume L, Δ, and Γ satisfy the hypotheses of the previous section. Let D be the determinant of the coefficient matrix and ω the distance between its eigenvalues,

$$D := a_{11}a_{22} - a_{12}a_{21}, \qquad \omega := \sqrt{(a_{22} - a_{11})^2 + 4a_{12}a_{21}},$$

and assume that the coefficient matrix (a_{ij}) satisfies

Hypothesis H_1.

(i) $0 < D < \frac{1}{4}(a_{11} + a_{22})^2$,

(ii) $a_{12} > 0$,

(iii) $a_{11} + a_{22} > 0$.

The coefficient matrix has eigenvalues

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}[a_{11} + a_{22} + \omega] \\ \lambda_2 &= \tfrac{1}{2}[a_{11} + a_{22} - \omega] \end{aligned} \tag{19}$$

with corresponding left eigenvectors

$$\begin{aligned} \Upsilon_1 &:= [Q_1, 1] \\ \Upsilon_2 &:= [Q_2, 1] \end{aligned} \tag{20}$$

where

$$\begin{aligned} Q_1 &= \tfrac{1}{2}[a_{11} - a_{22} + \omega]/a_{12} \\ Q_2 &= \tfrac{1}{2}[a_{11} - a_{22} - \omega]/a_{12}. \end{aligned} \tag{21}$$

The eigenvalues are positive and $Q_1 > Q_2$. Note that $Q_1 Q_2 = -a_{21}/a_{12}$ so that $Q_2 \leq 0 \leq Q_1$ whenever $a_{12} \geq 0$. Concerning the functions f_i we will assume

Hypothesis H_2. The measurable functions f_i defined on $\overline{\Omega} \times [0,\infty) \times \mathbf{R}^2$ are bounded on compact sets and are Lipschitz continuous in (t, u, v) and

$$uf_i(x,t,u,v) \geq 0 \qquad \forall i.$$

We will also assume the following "balance law":

Hypothesis H_3. There exist measurable functions α and β defined on Ω and $\partial\Omega$, respectively, such that for all $(u, v) \in \mathbf{R}^2$ we have

$$\begin{aligned} &-\lambda_2 \alpha(x) + Q_2 |f_1(x,t,u,v)| \\ &\qquad \leq |f_2(x,t,u,v)| \leq Q_1 |f_1(x,t,u,v)| + \lambda_1 \alpha(x), \end{aligned} \tag{22}$$

$$\begin{aligned} &-\beta(x) + Q_2 |f_3(x,t,u,v)| \\ &\qquad \leq |f_4(x,u,v)| \leq Q_1 |f_3(x,t,u,v)| + \beta(x). \end{aligned} \tag{23}$$

Hypothesis H_3 is of course satisfied (with $\alpha \equiv 0$) if we have a strict balance law $f_1 \equiv f_2$ and $f_3 \equiv f_4$ and $Q_2 \leq 1 \leq Q_1$. This last inequality is satisfied when $a_{21}^2 \geq a_{21}(a_{12} + a_{22} - a_{11})$.

Using the notation $U := [u, v]^T$, we define the sets

$$\mathscr{P}_{J,K} := \left\{ U \in \mathbf{R}^2 : J - \phi \leq \Upsilon_1 U, \Upsilon_2 U \leq K + \phi \right\}, \tag{24}$$

where ϕ is a strong solution of the elliptic problem

$$L\phi \geq \alpha \qquad \text{in } \Omega \tag{25}$$

$$\frac{\partial \phi}{\partial \nu} \geq \beta \qquad \text{on } \Delta. \tag{26}$$

By this we mean that if A is the bilinear functional associated with L (see Eq. (9) with $k = 1$), then

$$A(\phi, v) \geq (\alpha, v) + (\beta, \phi)_\Delta \qquad \forall \quad 0 \leq v \in H_\Delta^1(\Omega). \tag{27}$$

We may for example try to obtain ϕ by turning the inequalities into equalities and imposing a Dirichlet boundary condition on Γ. If $\alpha \in L^2(\Omega)$ (or, more generally, $\alpha \in H^{-1}(\Omega)$) and $\beta \in L^2(\partial\Omega)$ (or, more generally, $\beta \in H^{-1/2}(\partial\Omega)$) then a solution may be found (see [2, Chap. 8]). In the case of Neumann boundary conditions on all of $\partial\Omega$ it is of course necessary to satisfy the compatibility condition

$$\int_\Omega \alpha(x)\,dx \le -\int_{\partial\Omega} \beta(x)\,dS.$$

We shall simply assume the following.

Hypothesis H_4. There exists a function $\phi \in L^\infty(\Omega) \cap H^1(\Omega)$ that satisfies Eqs. (25), (26).

There is no restriction on the values of J and K so that if the initial conditions are bounded measurable functions then we can find constants J and K such that $U(x,0) \in \mathscr{P}_{J,K}$ for all $x \in \Omega$. Make the change of variables $W = \Upsilon U$ where Υ is the matrix

$$\Upsilon = \begin{pmatrix} Q_1 & 1 \\ Q_2 & 1 \end{pmatrix}. \tag{28}$$

Using Eq. (16) and letting $q := 1/(Q_1 - Q_2)$ we obtain

$$\begin{aligned} \frac{\partial w_1}{\partial t} &= \lambda_1 \Delta w_1 - Q_1 f_1(x,t,q(w_1 - w_2),v) \\ &\qquad + f_2(x,t,q(w_1 - w_2),v) \\ \frac{\partial w_2}{\partial t} &= \lambda_2 \Delta w_2 - Q_2 f_1(x,t,q(w_1 - w_2),v) \\ &\qquad + f_2(x,t,q(w_1 - w_2),v), \end{aligned} \tag{29}$$

where $v = q(Q_1 w_2 - Q_2 w_1)$. The boundary conditions become

$$\begin{aligned} \frac{\partial w_1}{\partial \nu} &= -Q_1 f_3(x,t,q(w_1 - w_2),v) + f_4(x,t,q(w_1 - w_2),v) \\ \frac{\partial w_2}{\partial \nu} &= -Q_2 f_3(x,t,q(w_1 - w_2),v) + f_4(x,t,q(w_1 - w_2),v). \end{aligned} \tag{30}$$

Let us apply Theorem 1 with $\Phi = (J - \phi, J - \phi)$ and $\Psi = (K + \phi, K + \phi)$ to show that this system has the invariant set

$$\mathscr{R}_{J,K} := \Upsilon^{-1}\mathscr{P}_{J,K} = \left\{ W \in \mathbf{R}^2 : J - \phi \le w_1, w_2 \le K + \phi \right\}.$$

To prove this we need to verify two inequalities along each of the four edges. For example, if we set $w_1 = J - \phi$ then we need to verify that

$$-\lambda_1 L\phi \leq -Q_1 f_1(x,t,q(J-\phi-w_2),v) + f_2(x,t,q(J-\phi-w_2),v),$$

for $J - \phi \leq w_2 \leq K + \phi$. Using the fact that the third argument in f_1 and f_2, i.e., u, is negative, we can write the right-hand side as $Q_1|f_1| - |f_2|$, which by Hypothesis H_3 is $\geq -\lambda_1 \alpha$. Hence the desired inequality follows from our choice of ϕ. The required inequality on the boundary condition follows analogously. Next we check the edge $w_1 = K + \phi$. We need

$$\lambda_1 L\phi \geq -Q_1 f_1(x,t,q(K+\phi-w_2),v) + f_2(x,t,q(K+\phi-w_2),v),$$

for $J - \phi \leq w_2 \leq K + \phi$. Now the third argument is positive so that the right-hand side may be written as $-Q_1|f_1| + |f_2|$ which is $\leq \lambda_1 \alpha \leq \lambda_1 L\phi$. Again the inequality for the boundary condition follows similarly. Next we set $w_2 = J - \phi$. We must show that

$$-\lambda_2 L\phi \leq -Q_2 f_1(x,t,q(w_1-J+\phi),v) + f_2(x,t,q(w_1-J+\phi),v),$$

for $J - \phi \leq w_1 \leq K + \phi$. The third argument is positive so that the right-hand side may be written as $-Q_2|f_1| + |f_2|$, which is $\geq -\lambda_2 \alpha \geq -\lambda_2 L\phi$. Finally, on the edge $w_2 = K + \phi$ we have

$$\begin{aligned}\lambda_2 L\phi \geq \lambda_2 \alpha \geq -Q_2 f_1&(x,t,q(w_1-K-\phi),v)\\ &+ f_2(x,t,q(w_1-K-\phi),v),\end{aligned}$$

for $J - \phi \leq w_1 \leq K + \phi$, since the third argument in the right-hand side is now negative so that it may be written as $Q_2|f_1| - |f_2|$. We have therefore proved the first part of the following.

THEOREM 2. *Suppose Ω is a bounded domain in $\mathbf{R}^n$ that is of class C^2 and suppose that Hypotheses H_1–H_4 are satisfied and f_3 and f_4 are independent of t. For any initial conditions $u(\cdot,0) \in L^\infty(\Omega)$ and $v(\cdot,0) \in L^\infty(\Omega)$ there exists a unique strong solution. This solution remains bounded since it will only attain values (a.e.) in the set $\mathscr{P}_{J,K}$ with*

$$J := \inf\{\min(Q_1 u(x,0) + v(x,0), Q_2 u(x,0) + v(x,0)) + \phi(x);\ x \in \Omega \cup \Gamma\},$$

$$K := \sup\{\max(Q_1 u(x,0) + v(x,0), Q_2 u(x,0) + v(x,0)) - \phi(x);\ x \in \Omega \cup \Gamma\},$$

where $u(x,0)$ is defined to be equal to the imposed Dirichlet boundary condition on Γ. In particular, any local classical solution with initial condi-

tions (u^0, v^0) *can be extended to a global solution and*

$$\|u(\cdot,t)\|_\infty \le \frac{2Ka_{12}}{\omega} \qquad \text{and} \qquad \|v(\cdot,t)\|_\infty \le \frac{K\max(\omega, |a_{11}-a_{22}|)}{\omega}, \tag{31}$$

where

$$-J = K = \|v^0\|_\infty + \big(\omega + |a_{11}-a_{22}|\big)\|u^0\|_\infty + \|\phi\|_\infty. \tag{32}$$

Proof. The only thing left to prove is the a priori bound. We observe that

$$\max(Q_1, |Q_2|) = \omega + |a_{11} - a_{22}|.$$

Therefore the initial condition for the new variables $(w_1(x,0), w_2(x,0)) \in \mathscr{R}_{-K,K}$ for almost all $x \in \overline{\Omega}$. It follows that for each $t > 0$ we have $(u(x,t), v(x,t)) \in \mathscr{P}_{-K,K}$. Since $u = q(w_1 - w_2)$ and $v = q(Q_1 w_2 - Q_2 w_1)$ we see that $|u| \le 2Kq = 2Kq_{12}/\omega$ and $|v| \le (Q_1 + |Q_2|)Kq = \max(|a_{11} - a_{22}|, \omega)K/\omega$.

Theorem 2 generalizes the result in [3] and [4] in several ways. It is easily verified that the hypotheses we place on the coefficients are weaker. In particular, we have eliminated the need for the difficult-to-establish bound on $|a_{22} - a_{11} - a_{12} + a_{21}|$. Also, we allow for more general nonlinearities that need not satisfy a polynomial growth condition and do not need to satisfy a strict balance law. One might consider generalizing the above method to quasilinear systems by allowing nonlinear changes of dependent variables. Unfortunately, such a change of variables brings in quadratic nonlinearities in the first-order derivatives that are generally difficult to cope with.

Remark. We can relax Hypothesis H_3 by requiring that it hold only when $|u| + |v|$ is large. The functions α and β can sometimes play a significant role, as illustrated by the following.

EXAMPLE. Suppose that $a_{22} > a_{11}$ or that $a_{21} > 0$. Assume that f_3 and f_4 are continuous functions with the only restriction being that f_4 is bounded. Also, suppose that we have

$$|f_1(x,t,u,v)|/(|u|+|v|) \to \infty \qquad \text{as } |u|+|v| \to \infty$$

and

$$|f_2(x,t,u,v)|/f_1(x,t,u,v) \to \infty \qquad \text{as } |u|+|v| \to \infty$$

uniformly in (x,t). Let β be a bound for $|f_4|$; then since $Q_2 < 0 < Q_1$ we see that the inequalities (23) will be satisfied. Now pick the constant α so that the compatibility condition is satisfied. Note that $\alpha \leq 0$. Nevertheless, because of the above assumed growth conditions it is clear that (22) will be satisfied provided that $|u| + |v|$ is sufficiently large. Hence bounded initial conditions lead to bounded solutions.

It is important to note that our method is contingent on the diffusion coefficients of the transformed system (i.e., the eigenvalues of the coefficient matrix) not being equal. This is underscored by the presence of ω $(= |\lambda_1 - \lambda_2|)$ in the denominator of the a priori bounds (31).

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